Appendix A

Here we provide the reparameterization for the LGM with random knots, random floors, as well as random ceilings following the steps discussed by Preacher and Hancock (2015).

1) (Re)parameterize the target function to contain substantively important parameters

$$y = f(\gamma_1, \gamma_2, y_f, y_c, t) = \frac{1}{2}\left[y_f + y_c + \left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\sqrt{(\gamma_1 - t)^2} - \left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\sqrt{(t - \gamma_2)^2})\right]$$

2) Linearize the target function to render it specifiable using SEM

$$\tilde{y} = f(\gamma_1, \gamma_2, y_f, y_c, t)|_{\mu_1,\mu_2,\mu_f,\mu_c} + (\gamma_1 - \mu_1)\frac{\partial f}{\partial \gamma_1}\bigg|_{\mu_1,\mu_2,\mu_f,\mu_c} + (\gamma_2 - \mu_2)\frac{\partial f}{\partial \gamma_2}\bigg|_{\mu_1,\mu_2,\mu_f,\mu_c}$$

$$+ (y_f - \mu_f)\frac{\partial f}{\partial y_f}\bigg|_{\mu_1,\mu_2,\mu_f,\mu_c} + (y_c - \mu_c)\frac{\partial f}{\partial y_c}\bigg|_{\mu_1,\mu_2,\mu_f,\mu_c},$$

where

$$\frac{\partial f}{\partial \gamma_1}\bigg|_{\mu_1,\mu_2,\mu_f,\mu_c} = \frac{1}{2}\left[\frac{\mu_c - \mu_f}{\mu_2 - \mu_1}\right]\left\{(\mu_2 - \mu_1)^{-1}\left[\sqrt{(\mu_1 - t)^2} - \sqrt{(t - \mu_2)^2}\right] + \frac{\mu_1 - t}{\sqrt{(\mu_1 - t)^2}}\right\}$$

$$\frac{\partial f}{\partial \gamma_2}\bigg|_{\mu_1,\mu_2,\mu_f,\mu_c} = \frac{1}{2}\left[\frac{\mu_c - \mu_f}{\mu_2 - \mu_1}\right]\left\{(\mu_2 - \mu_1)^{-1}\left[\sqrt{(t - \mu_2)^2} - \sqrt{(\mu_1 - t)^2}\right] + \frac{t - \mu_2}{\sqrt{(\mu_2 - t)^2}}\right\}$$

$$\frac{\partial f}{\partial y_f}\bigg|_{\mu_1,\mu_2,\mu_f,\mu_c} = \frac{1}{2}\left\{1 + (\mu_2 - \mu_1)^{-1}\left[\sqrt{(t - \mu_2)^2} - \sqrt{(\mu_1 - t)^2}\right]\right\}$$

$$\frac{\partial f}{\partial y_c}\bigg|_{\mu_1,\mu_2,\mu_f,\mu_c} = \frac{1}{2}\left\{1 + (\mu_2 - \mu_1)^{-1}\left[\sqrt{(\mu_1 - t)^2} - \sqrt{(t - \mu_2)^2}\right]\right\}.$$

3) Specify the model using the structured latent curve modeling (SLCM) based approach

The linearized function in matrix form is:

$$\mathbf{Y}_i = \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i,$$

where

$$\boldsymbol{\tau} = f(\gamma_1, \gamma_2, y_f, y_c, \mathbf{t})|_{\mu_1, \mu_2, \mu_f, \mu_c} = \begin{bmatrix} \frac{1}{2}\left\{\mu_f + \mu_c + \left[\frac{\mu_c - \mu_f}{\mu_2 - \mu_1}\right]\sqrt{(\mu_1 - 0)^2} - \left[\frac{\mu_c - \mu_f}{\mu_2 - \mu_1}\right]\sqrt{(0 - \mu_2)^2})\right\} \\ \frac{1}{2}\left\{\mu_f + \mu_c + \left[\frac{\mu_c - \mu_f}{\mu_2 - \mu_1}\right]\sqrt{(\mu_1 - 1)^2} - \left[\frac{\mu_c - \mu_f}{\mu_2 - \mu_1}\right]\sqrt{(1 - \mu_2)^2})\right\} \\ \vdots \\ \frac{1}{2}\left\{\mu_f + \mu_c + \left[\frac{\mu_c - \mu_f}{\mu_2 - \mu_1}\right]\sqrt{(\mu_1 - T)^2} - \left[\frac{\mu_c - \mu_f}{\mu_2 - \mu_1}\right]\sqrt{(T - \mu_2)^2})\right\} \end{bmatrix},$$

$$\boldsymbol{\Lambda}$$

$$= \left[\begin{array}{c|c|c|c} \frac{1}{2}\left[\frac{\mu_c - \mu_f}{\mu_2 - \mu_1}\right]\left\{(\mu_2 - \mu_1)^{-1}\left[\sqrt{(\mu_1 - 0)^2} - \sqrt{(0 - \mu_2)^2}\right] + \frac{\mu_1 - 0}{\sqrt{(\mu_1 - 0)^2}}\right\} & \frac{1}{2}\left[\frac{\mu_c - \mu_f}{\mu_2 - \mu_1}\right]\left\{(\mu_2 - \mu_1)^{-1}\left[\sqrt{(0 - \mu_2)^2} - \sqrt{(\mu_1 - 0)^2}\right] + \frac{0 - \mu_2}{\sqrt{(\mu_2 - 0)^2}}\right\} & \frac{1}{2}\left\{1 + \frac{1}{\mu_2 - \mu_1}\left[\sqrt{(0 - \mu_2)^2} - \sqrt{(\mu_1 - 0)^2}\right]\right\} & \frac{1}{2}\left\{1 + \frac{1}{\mu_2 - \mu_1}\left[\sqrt{(\mu_1 - 0)^2} - \sqrt{(0 - \mu_2)^2}\right]\right\} \\ \frac{1}{2}\left[\frac{\mu_c - \mu_f}{\mu_2 - \mu_1}\right]\left\{(\mu_2 - \mu_1)^{-1}\left[\sqrt{(\mu_1 - 1)^2} - \sqrt{(1 - \mu_2)^2}\right] + \frac{\mu_1 - 1}{\sqrt{(\mu_1 - 1)^2}}\right\} & \frac{1}{2}\left[\frac{\mu_c - \mu_f}{\mu_2 - \mu_1}\right]\left\{(\mu_2 - \mu_1)^{-1}\left[\sqrt{(1 - \mu_2)^2} - \sqrt{(\mu_1 - 1)^2}\right] + \frac{1 - \mu_2}{\sqrt{(\mu_2 - 1)^2}}\right\} & \frac{1}{2}\left\{1 + \frac{1}{\mu_2 - \mu_1}\left[\sqrt{(1 - \mu_2)^2} - \sqrt{(\mu_1 - 1)^2}\right]\right\} & \frac{1}{2}\left\{1 + \frac{1}{\mu_2 - \mu_1}\left[\sqrt{(\mu_1 - 1)^2} - \sqrt{(1 - \mu_2)^2}\right]\right\} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2}\left[\frac{\mu_c - \mu_f}{\mu_2 - \mu_1}\right]\left\{(\mu_2 - \mu_1)^{-1}\left[\sqrt{(\mu_1 - T)^2} - \sqrt{(T - \mu_2)^2}\right] + \frac{\mu_1 - T}{\sqrt{(\mu_1 - T)^2}}\right\} & \frac{1}{2}\left[\frac{\mu_c - \mu_f}{\mu_2 - \mu_1}\right]\left\{(\mu_2 - \mu_1)^{-1}\left[\sqrt{(T - \mu_2)^2} - \sqrt{(\mu_1 - T)^2}\right] + \frac{T - \mu_2}{\sqrt{(\mu_2 - T)^2}}\right\} & \frac{1}{2}\left\{1 + \frac{1}{\mu_2 - \mu_1}\left[\sqrt{(T - \mu_2)^2} - \sqrt{(\mu_1 - T)^2}\right]\right\} & \frac{1}{2}\left\{1 + \frac{1}{\mu_2 - \mu_1}\left[\sqrt{(\mu_1 - T)^2} - \sqrt{(T - \mu_2)^2}\right]\right\} \end{array}\right]$$

and

$$\boldsymbol{\eta}_i = \begin{bmatrix} \gamma_{1i} - \mu_1 \\ \gamma_{2i} - \mu_2 \\ y_{fi} - \mu_f \\ y_{ci} - \mu_c \end{bmatrix}.$$

Appendix B

Here we provide the derivation of Eq. (7a) and Eq. (7b), which are the first order partial derivatives of the target function with respect to γ_1 and γ_2, respectively.

First, the following partial first derivatives can be obtained using elementary calculus:

$$\frac{\partial}{\partial\gamma_1}\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right] = \frac{y_c - y_f}{(\gamma_2 - \gamma_1)^2}$$

$$\frac{\partial}{\partial\gamma_2}\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right] = -\frac{y_c - y_f}{(\gamma_2 - \gamma_1)^2}$$

$$\frac{\partial}{\partial\gamma_1}\left[\sqrt{(\gamma_1 - t)^2}\right] = \frac{\gamma_1 - t}{\sqrt{(\gamma_1 - t)^2}}$$

$$\frac{\partial}{\partial\gamma_2}\left[\sqrt{(t - \gamma_2)^2}\right] = \frac{\gamma_2 - t}{\sqrt{(\gamma_2 - t)^2}}.$$

The first order partial derivatives of the target function with respect to γ_1 and γ_2 can thus be calculated with the basic differentiation rules as follows:

$$\frac{\partial f}{\partial\gamma_1} = \frac{1}{2}\left(0 + 0 + \frac{\partial}{\partial\gamma_1}\left\{\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\sqrt{(\gamma_1 - t)^2}\right\} - \frac{\partial}{\partial\gamma_1}\left\{\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\sqrt{(t - \gamma_2)^2}\right\}\right)$$

$$= \frac{1}{2}\left(\left\{\frac{\partial}{\partial\gamma_1}\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\right\}\sqrt{(\gamma_1 - t)^2} + \left\{\frac{\partial}{\partial\gamma_1}\sqrt{(\gamma_1 - t)^2}\right\}\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right] - \left\{\frac{\partial}{\partial\gamma_1}\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\right\}\sqrt{(t - \gamma_2)^2}\right)$$

$$= \frac{1}{2}\left\{\frac{y_c - y_f}{(\gamma_2 - \gamma_1)^2}\sqrt{(\gamma_1 - t)^2} + \frac{\gamma_1 - t}{\sqrt{(\gamma_1 - t)^2}}\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right] - \frac{y_c - y_f}{(\gamma_2 - \gamma_1)^2}\sqrt{(t - \gamma_2)^2}\right\}$$

$$= \frac{1}{2}\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\left\{\frac{\sqrt{(\gamma_1 - t)^2}}{\gamma_2 - \gamma_1} + \frac{\gamma_1 - t}{\sqrt{(\gamma_1 - t)^2}} - \frac{\sqrt{(t - \gamma_2)^2}}{\gamma_2 - \gamma_1}\right\}$$

$$= \frac{1}{2}\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\left\{(\gamma_2 - \gamma_1)^{-1}\left[\sqrt{(\gamma_1 - t)^2} - \sqrt{(t - \gamma_2)^2}\right] + \frac{\gamma_1 - t}{\sqrt{(\gamma_1 - t)^2}}\right\}$$

and

$$\frac{\partial f}{\partial \gamma_2} = \frac{1}{2}\left(0 + 0 + \frac{\partial}{\partial \gamma_2}\left\{\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\sqrt{(\gamma_1 - t)^2}\right\} - \frac{\partial}{\partial \gamma_2}\left\{\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\sqrt{(t - \gamma_2)^2}\right\}\right)$$

$$= \frac{1}{2}\left(\left\{\frac{\partial}{\partial \gamma_2}\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\right\}\sqrt{(\gamma_1 - t)^2} - \left\{\frac{\partial}{\partial \gamma_2}\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\right\}\sqrt{(t - \gamma_2)^2} - \left\{\frac{\partial}{\partial \gamma_2}\left[\sqrt{(t - \gamma_2)^2}\right]\right\}\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\right)$$

$$= \frac{1}{2}\left\{-\frac{y_c - y_f}{(\gamma_2 - \gamma_1)^2}\sqrt{(\gamma_1 - t)^2} + \frac{y_c - y_f}{(\gamma_2 - \gamma_1)^2}\sqrt{(t - \gamma_2)^2} - \frac{\gamma_2 - t}{\sqrt{(\gamma_2 - t)^2}}\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\right\}$$

$$= \frac{1}{2}\left[\frac{y_c - y_f}{\gamma_2 - \gamma_1}\right]\left\{(\gamma_2 - \gamma_1)^{-1}\left[\sqrt{(t - \gamma_2)^2} - \sqrt{(\gamma_1 - t)^2}\right] + \frac{t - \gamma_2}{\sqrt{(\gamma_2 - t)^2}}\right\}.$$

Therefore, the first order partial derivatives evaluated at the population means of γ_1 and γ_2 are

$$\left.\frac{\partial f}{\partial \gamma_1}\right|_{\mu_1,\mu_2} = \frac{1}{2}\left[\frac{y_c - y_f}{\mu_2 - \mu_1}\right]\left\{(\mu_2 - \mu_1)^{-1}\left[\sqrt{(\mu_1 - t)^2} - \sqrt{(t - \mu_2)^2}\right] + \frac{\mu_1 - t}{\sqrt{(\mu_1 - t)^2}}\right\}$$

$$\left.\frac{\partial f}{\partial \gamma_2}\right|_{\mu_1,\mu_2} = \frac{1}{2}\left[\frac{y_c - y_f}{\mu_2 - \mu_1}\right]\left\{(\mu_2 - \mu_1)^{-1}\left[\sqrt{(t - \mu_2)^2} - \sqrt{(\mu_1 - t)^2}\right] + \frac{t - \mu_2}{\sqrt{(\mu_2 - t)^2}}\right\}.$$

Appendix C

Here we demonstrate that the mean trajectory $\boldsymbol{\tau}$ for values of *t* below μ_1 are equal to y_f, and for value of *t* above μ_2 are equal to y_c.

When $t < \mu_1$,

$$\boldsymbol{\tau} = f(\gamma_1, \gamma_2, t)|_{\mu_1, \mu_2} = \frac{1}{2}\left\{y_f + y_c + \left[\frac{y_c - y_f}{\mu_2 - \mu_1}\right]\sqrt{(\mu_1 - t)^2} - \left[\frac{y_c - y_f}{\mu_2 - \mu_1}\right]\sqrt{(t - \mu_2)^2}\right\}$$

$$= \frac{1}{2}\left\{y_f + y_c + \left[\frac{y_c - y_f}{\mu_2 - \mu_1}\right](\mu_1 - t) - \left[\frac{y_c - y_f}{\mu_2 - \mu_1}\right](\mu_2 - t))\right\}$$

$$= \frac{1}{2}\left\{y_f + y_c + \left[\frac{y_c - y_f}{\mu_2 - \mu_1}\right](\mu_1 - t - \mu_2 + t)\right\}$$

$$= \frac{1}{2}\left\{y_f + y_c + \left[\frac{y_c - y_f}{\mu_2 - \mu_1}\right](\mu_1 - \mu_2)\right\}$$

$$= \frac{1}{2}\{y_f + y_c + y_f - y_c\}$$

$$= y_f.$$

When $t > \gamma_2$,

$$\boldsymbol{\tau} = f(\gamma_1, \gamma_2, t)|_{\mu_1, \mu_2} = \frac{1}{2}\left\{y_f + y_c + \left[\frac{y_c - y_f}{\mu_2 - \mu_1}\right]\sqrt{(\mu_1 - t)^2} - \left[\frac{y_c - y_f}{\mu_2 - \mu_1}\right]\sqrt{(t - \mu_2)^2}\right\}$$

$$= \frac{1}{2}\left\{y_f + y_c + \left[\frac{y_c - y_f}{\mu_2 - \mu_1}\right](t - \mu_1) - \left[\frac{y_c - y_f}{\mu_2 - \mu_1}\right](t - \mu_2)\right\}$$

$$= \frac{1}{2}\left\{y_f + y_c + \left[\frac{y_c - y_f}{\mu_2 - \mu_1}\right](t - \mu_1 - t + \mu_2)\right\}$$

$$= \frac{1}{2}\left\{y_f + y_c + \left[\frac{y_c - y_f}{\mu_2 - \mu_1}\right](\mu_2 - \mu_1)\right\}$$

$$= \frac{1}{2}\{y_f + y_c + y_c - y_f\}$$

$$= y_c.$$

Appendix D

1) M*plus* code used in illustrative example 1:

```
TITLE: Illustrative Example 1

DATA: FILE IS Censored-T5-v2.csv;

Variable:
NAMES ARE week1 week2 week3 week4 week5 ;

Usevariables ARE
week1 week2 week3 week4 week5 ;

CENSORED ARE  week1-week2 (b) week4-week5 (a);


ANALYSIS:
ESTIMATOR=WLSMV;

MODEL:
!Factor Loadings
g1 BY week1*(g1t1) !free the first loading
week2-week5(g1t2-g1t5);
g2 BY week1*(g2t1) !free the first loading
week2-week5(g2t2-g2t5);
!Intercepts of each individual measurement
[week1-week5](tau1-tau5);
!Factor Means
[g1@0]; [g2@0];
!Variances and covariances
week1-week5 (r); !constrain the error variances to be equal across time
points
g1 g2;
g1 WITH g2;


MODEL CONSTRAINT:

NEW (mu_g1*0.9); !mean of the first random knot
NEW (mu_g2*3.1); !mean of the second random knot

!The intercepts constraints
tau1=1/2*(0+5+((5-0)/(mu_g2-mu_g1))*sqrt((mu_g1-0)^2)
-((5-0)/(mu_g2-mu_g1))*sqrt((0-mu_g2)^2));
tau2=1/2*(0+5+((5-0)/(mu_g2-mu_g1))*sqrt((mu_g1-1)^2)
-((5-0)/(mu_g2-mu_g1))*sqrt((1-mu_g2)^2));
tau3=1/2*(0+5+((5-0)/(mu_g2-mu_g1))*sqrt((mu_g1-2)^2)
-((5-0)/(mu_g2-mu_g1))*sqrt((2-mu_g2)^2));
tau4=1/2*(0+5+((5-0)/(mu_g2-mu_g1))*sqrt((mu_g1-3)^2)
-((5-0)/(mu_g2-mu_g1))*sqrt((3-mu_g2)^2));
tau5=1/2*(0+5+((5-0)/(mu_g2-mu_g1))*sqrt((mu_g1-4)^2)
-((5-0)/(mu_g2-mu_g1))*sqrt((4-mu_g2)^2));
```

```
  !loadings on g1 (gamma1, the first random knot)
  g1t1=(1/2)*((5-0)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
  (sqrt((mu_g1-0)^2)-sqrt((0-mu_g2)^2))+((mu_g1-0)/sqrt((mu_g1-0)^2)));
  g1t2=(1/2)*((5-0)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
  (sqrt((mu_g1-1)^2)-sqrt((1-mu_g2)^2))+((mu_g1-1)/sqrt((mu_g1-1)^2)));
  g1t3=(1/2)*((5-0)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
  (sqrt((mu_g1-2)^2)-sqrt((2-mu_g2)^2))+((mu_g1-2)/sqrt((mu_g1-2)^2)));
  g1t4=(1/2)*((5-0)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
  (sqrt((mu_g1-3)^2)-sqrt((3-mu_g2)^2))+((mu_g1-3)/sqrt((mu_g1-3)^2)));
  g1t5=(1/2)*((5-0)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
  (sqrt((mu_g1-4)^2)-sqrt((4-mu_g2)^2))+((mu_g1-4)/sqrt((mu_g1-4)^2)));


  !loadings on g2 (gamma2) the second random knot
  g2t1=1/2*((5-0)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
  (sqrt((0-mu_g2)^2)-sqrt((mu_g1-0)^2))+((0-mu_g2)/sqrt((mu_g2-0)^2)));
  g2t2=1/2*((5-0)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
  (sqrt((1-mu_g2)^2)-sqrt((mu_g1-1)^2))+((1-mu_g2)/sqrt((mu_g2-1)^2)));
  g2t3=1/2*((5-0)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
  (sqrt((2-mu_g2)^2)-sqrt((mu_g1-2)^2))+((2-mu_g2)/sqrt((mu_g2-2)^2)));
  g2t4=1/2*((5-0)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
  (sqrt((3-mu_g2)^2)-sqrt((mu_g1-3)^2))+((3-mu_g2)/sqrt((mu_g2-3)^2)));
  g2t5=1/2*((5-0)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
  (sqrt((4-mu_g2)^2)-sqrt((mu_g1-4)^2))+((4-mu_g2)/sqrt((mu_g2-4)^2)));


    Output:
    sampstat standardized RESIDUAL;
```

2) M*plus* code used in illustrative example 2:

```
TITLE: The second illustrative example:
       children's reading comprehension scores over 10 measurement occasions.
       A model with random floors, random ceilings, and random knots is
fitted to the data.

DATA: FILE IS wjpcw_gender.csv;

DEFINE: CENTER gender (GRANDMEAN);

Variable:
NAMES ARE ID
y2  y3  y4  y5  y6  y7  y8
y9  y0  y1 gender
;

Usevariables ARE
y0 y1 y2 y3 y4 y5 y6 y7 y8 y9 gender
;


! no censored variables given the PC W-scores are IRT-based scores;
! they are continuous without a "hard" floor or "ceiling";
```

```
MISSING ARE ALL (999);

ANALYSIS:
COVERAGE=0;

MODEL:
!Factor Loadings
g1 BY y0*(g1t1) !First random knot
y1-y9(g1t2-g1t10);
g2 BY y0*(g2t1) !Second random knot
y1-y9(g2t2-g2t10);
fl BY y0*(ft1) !Random floors
y1-y9(ft2-ft10);
cl BY y0*(ct1) !Random ceilings
y1-y9(ct2-ct10);

g1 ON gender;

g2 WITH gender@0;
fl WITH gender@0;
cl WITH gender@0;

[y0-y9] (tau1-tau10);
[g1@0]; [g2@0]; [fl@0]; [cl@0];
[gender@0];

!Variances and covariances
y0-y9;
g1 g2 fl;
cl;
g1 WITH g2;
g1 WITH fl;
g2 WITH fl;
g1 WITH cl (v);
g2 WITH cl (v);
fl WITH cl;


MODEL CONSTRAINT:
 NEW (mu_g1*1.5); !mean of the first random knot
 NEW (mu_g2*8.5); !mean of the second random knot
 NEW (mu_fl*300); !mean of the floors
 NEW (mu_cl*500); !mean of the ceilings

! intercept, tau
tau1=1/2*(mu_fl+mu_cl+((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((mu_g1-0)^2)-
((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((0-mu_g2)^2));

tau2=1/2*(mu_fl+mu_cl+((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((mu_g1-1)^2)-
((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((1-mu_g2)^2));
tau3=1/2*(mu_fl+mu_cl+((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((mu_g1-2)^2)-
((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((2-mu_g2)^2));
tau4=1/2*(mu_fl+mu_cl+((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((mu_g1-3)^2)-
((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((3-mu_g2)^2));
tau5=1/2*(mu_fl+mu_cl+((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((mu_g1-4)^2)-
((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((4-mu_g2)^2));
```

```
tau6=1/2*(mu_fl+mu_cl+((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((mu_g1-5)^2)-
((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((5-mu_g2)^2));
tau7=1/2*(mu_fl+mu_cl+((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((mu_g1-6)^2)-
((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((6-mu_g2)^2));
tau8=1/2*(mu_fl+mu_cl+((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((mu_g1-7)^2)-
((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((7-mu_g2)^2));
tau9=1/2*(mu_fl+mu_cl+((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((mu_g1-8)^2)-
((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((8-mu_g2)^2));
tau10=1/2*(mu_fl+mu_cl+((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((mu_g1-9)^2)-
((mu_cl-mu_fl)/(mu_g2-mu_g1))*sqrt((9-mu_g2)^2));


!loadings on g1 (gamma1, the first random knot)
g1t1=(1/2)*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((mu_g1-
0)^2)
-sqrt((0-mu_g2)^2))+((mu_g1-0)/sqrt((mu_g1-0)^2)));
g1t2=(1/2)*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((mu_g1-
1)^2)
-sqrt((1-mu_g2)^2))+((mu_g1-1)/sqrt((mu_g1-1)^2)));
g1t3=(1/2)*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((mu_g1-
2)^2)
-sqrt((2-mu_g2)^2))+((mu_g1-2)/sqrt((mu_g1-2)^2)));
g1t4=(1/2)*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((mu_g1-
3)^2)
-sqrt((3-mu_g2)^2))+((mu_g1-3)/sqrt((mu_g1-3)^2)));
g1t5=(1/2)*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((mu_g1-
4)^2)
-sqrt((4-mu_g2)^2))+((mu_g1-4)/sqrt((mu_g1-4)^2)));
g1t6=(1/2)*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((mu_g1-
5)^2)
-sqrt((5-mu_g2)^2))+((mu_g1-5)/sqrt((mu_g1-5)^2)));
g1t7=(1/2)*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((mu_g1-
6)^2)
-sqrt((6-mu_g2)^2))+((mu_g1-6)/sqrt((mu_g1-6)^2)));
g1t8=(1/2)*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((mu_g1-
7)^2)
-sqrt((7-mu_g2)^2))+((mu_g1-7)/sqrt((mu_g1-7)^2)));
g1t9=(1/2)*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((mu_g1-
8)^2)
-sqrt((8-mu_g2)^2))+((mu_g1-8)/sqrt((mu_g1-8)^2)));
g1t10=(1/2)*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((mu_g1-
9)^2)
-sqrt((9-mu_g2)^2))+((mu_g1-9)/sqrt((mu_g1-9)^2)));



!loadings on g2 (gamma2) the second random knot
g2t1=1/2*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((0-mu_g2)^2)
-sqrt((mu_g1-0)^2))+((0-mu_g2)/sqrt((mu_g2-0)^2)));
g2t2=1/2*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((1-mu_g2)^2)
-sqrt((mu_g1-1)^2))+((1-mu_g2)/sqrt((mu_g2-1)^2)));
g2t3=1/2*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((2-mu_g2)^2)
-sqrt((mu_g1-2)^2))+((2-mu_g2)/sqrt((mu_g2-2)^2)));
g2t4=1/2*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((3-mu_g2)^2)
-sqrt((mu_g1-3)^2))+((3-mu_g2)/sqrt((mu_g2-3)^2)));
g2t5=1/2*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((4-mu_g2)^2)
-sqrt((mu_g1-4)^2))+((4-mu_g2)/sqrt((mu_g2-4)^2)));
```

```
g2t6=1/2*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((5-mu_g2)^2)
-sqrt((mu_g1-5)^2))+((5-mu_g2)/sqrt((mu_g2-5)^2)));
g2t7=1/2*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((6-mu_g2)^2)
-sqrt((mu_g1-6)^2))+((6-mu_g2)/sqrt((mu_g2-6)^2)));
g2t8=1/2*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((7-mu_g2)^2)
-sqrt((mu_g1-7)^2))+((7-mu_g2)/sqrt((mu_g2-7)^2)));
g2t9=1/2*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((8-mu_g2)^2)
-sqrt((mu_g1-8)^2))+((8-mu_g2)/sqrt((mu_g2-8)^2)));
g2t10=1/2*((mu_cl-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*(sqrt((9-mu_g2)^2)
-sqrt((mu_g1-9)^2))+((9-mu_g2)/sqrt((mu_g2-9)^2)));


! loadings on the random floor latent factor
ft1=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((0-mu_g2)^2)-sqrt((mu_g1-0)^2)));
ft2=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((1-mu_g2)^2)-sqrt((mu_g1-1)^2)));
ft3=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((2-mu_g2)^2)-sqrt((mu_g1-2)^2)));
ft4=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((3-mu_g2)^2)-sqrt((mu_g1-3)^2)));
ft5=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((4-mu_g2)^2)-sqrt((mu_g1-4)^2)));
ft6=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((5-mu_g2)^2)-sqrt((mu_g1-5)^2)));
ft7=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((6-mu_g2)^2)-sqrt((mu_g1-6)^2)));
ft8=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((7-mu_g2)^2)-sqrt((mu_g1-7)^2)));
ft9=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((8-mu_g2)^2)-sqrt((mu_g1-8)^2)));
ft10=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((9-mu_g2)^2)-sqrt((mu_g1-9)^2)));


!loadings on the random ceiling latent factor
ct1=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((mu_g1-0)^2)-sqrt((0-mu_g2)^2)));
ct2=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((mu_g1-1)^2)-sqrt((1-mu_g2)^2)));
ct3=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((mu_g1-2)^2)-sqrt((2-mu_g2)^2)));
ct4=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((mu_g1-3)^2)-sqrt((3-mu_g2)^2)));
ct5=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((mu_g1-4)^2)-sqrt((4-mu_g2)^2)));
ct6=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((mu_g1-5)^2)-sqrt((5-mu_g2)^2)));
ct7=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((mu_g1-6)^2)-sqrt((6-mu_g2)^2)));
ct8=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((mu_g1-7)^2)-sqrt((7-mu_g2)^2)));
ct9=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((mu_g1-8)^2)-sqrt((8-mu_g2)^2)));
ct10=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((mu_g1-9)^2)-sqrt((9-mu_g2)^2)));


Output:
sampstat standardized RESIDUAL;
```

3) M*plus* code used to obtain the model fit statistics and scaling correction factor for the alternative model with random floors fitted to the simulated data in the first illustrative example:

```
TITLE: First illustrative example model comparison:
       the alternative model with random floors

DATA: FILE IS Censored-T5-v2.csv;
```

```
Variable:
NAMES ARE week1 week2 week3 week4 week5 ;

Usevariables ARE
week1 week2 week3 week4 week5 ;

CENSORED ARE  week1-week2 (b) week4-week5 (a);


ANALYSIS:
ESTIMATOR=WLSM; !use the WLSM estimator

MODEL:
! the first random knot
g1 BY week1*(g1t1) !free the first loading
week2-week5(g1t2-g1t5);
! the second random knot
g2 BY week1*(g2t1) !free the first loading
week2-week5(g2t2-g2t5);
! random floors
fl BY week1*(flt1) !free the first loading
week2-week5(flt2-flt5);

!Intercepts of each individual measurement
[week1-week5](tau1-tau5);

!Factor Means, all constrained to be zero
[g1@0]; [g2@0]; [fl@0];

!Variances and covariances
week1-week5 (r); !constrained to be homegeneous.
g1 g2 fl;
g1 WITH g2;
g1 WITH fl;
g2 WITH fl;


MODEL CONSTRAINT:

NEW (mu_g1*0.9);  !mean of the first random knot
NEW (mu_g2*3.1);  !mean of the second random knot
NEW (mu_fl*0);    !mean of the random floors


 !The intercepts constraints
tau1=1/2*(mu_fl+5+((5-mu_fl)/(mu_g2-mu_g1))*sqrt((mu_g1-0)^2)
-((5-mu_fl)/(mu_g2-mu_g1))*sqrt((0-mu_g2)^2));
tau2=1/2*(mu_fl+5+((5-mu_fl)/(mu_g2-mu_g1))*sqrt((mu_g1-1)^2)
-((5-mu_fl)/(mu_g2-mu_g1))*sqrt((1-mu_g2)^2));
tau3=1/2*(mu_fl+5+((5-mu_fl)/(mu_g2-mu_g1))*sqrt((mu_g1-2)^2)
-((5-mu_fl)/(mu_g2-mu_g1))*sqrt((2-mu_g2)^2));
tau4=1/2*(mu_fl+5+((5-mu_fl)/(mu_g2-mu_g1))*sqrt((mu_g1-3)^2)
-((5-mu_fl)/(mu_g2-mu_g1))*sqrt((3-mu_g2)^2));
tau5=1/2*(mu_fl+5+((5-mu_fl)/(mu_g2-mu_g1))*sqrt((mu_g1-4)^2)
-((5-mu_fl)/(mu_g2-mu_g1))*sqrt((4-mu_g2)^2));
```

```
  !loadings on g1 (gamma1, the first random knot)
g1t1=(1/2)*((5-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
(sqrt((mu_g1-0)^2)-sqrt((0-mu_g2)^2))+((mu_g1-0)/sqrt((mu_g1-0)^2)));
g1t2=(1/2)*((5-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
(sqrt((mu_g1-1)^2)-sqrt((1-mu_g2)^2))+((mu_g1-1)/sqrt((mu_g1-1)^2)));
g1t3=(1/2)*((5-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
(sqrt((mu_g1-2)^2)-sqrt((2-mu_g2)^2))+((mu_g1-2)/sqrt((mu_g1-2)^2)));
g1t4=(1/2)*((5-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
(sqrt((mu_g1-3)^2)-sqrt((3-mu_g2)^2))+((mu_g1-3)/sqrt((mu_g1-3)^2)));
g1t5=(1/2)*((5-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
(sqrt((mu_g1-4)^2)-sqrt((4-mu_g2)^2))+((mu_g1-4)/sqrt((mu_g1-4)^2)));

  !loadings on g2 (gamma2) the second random knot
g2t1=1/2*((5-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
(sqrt((0-mu_g2)^2)-sqrt((mu_g1-0)^2))+((0-mu_g2)/sqrt((mu_g2-0)^2)));
g2t2=1/2*((5-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
(sqrt((1-mu_g2)^2)-sqrt((mu_g1-1)^2))+((1-mu_g2)/sqrt((mu_g2-1)^2)));
g2t3=1/2*((5-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
(sqrt((2-mu_g2)^2)-sqrt((mu_g1-2)^2))+((2-mu_g2)/sqrt((mu_g2-2)^2)));
g2t4=1/2*((5-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
(sqrt((3-mu_g2)^2)-sqrt((mu_g1-3)^2))+((3-mu_g2)/sqrt((mu_g2-3)^2)));
g2t5=1/2*((5-mu_fl)/(mu_g2-mu_g1))*((1/(mu_g2-mu_g1))*
(sqrt((4-mu_g2)^2)-sqrt((mu_g1-4)^2))+((4-mu_g2)/sqrt((mu_g2-4)^2)));

! random floors
flt1=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((0-mu_g2)^2)-sqrt((mu_g1-0)^2)));
flt2=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((1-mu_g2)^2)-sqrt((mu_g1-1)^2)));
flt3=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((2-mu_g2)^2)-sqrt((mu_g1-2)^2)));
flt4=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((3-mu_g2)^2)-sqrt((mu_g1-3)^2)));
flt5=(1/2)*(1+(1/(mu_g2-mu_g1))*(sqrt((4-mu_g2)^2)-sqrt((mu_g1-4)^2)));

  Output:
  sampstat standardized RESIDUAL;
```

Appendix E

Here we provide the estimated intercept vector and estimated factor loadings matrix for the first illustrative example. It should be noted that the intercepts and factor loadings were not directly estimated as they were constrained to be functions of the directly estimated parameters—μ_1 and μ_2. Their standard errors are obtained by the Delta method by default (see MacIntosh & Hashim, 2003; Muthén & Asparouhov, 2002).

The estimated intercept vector (with standard error in parenthesis) is

$$\hat{\boldsymbol{\tau}} = \begin{bmatrix} 0.000\ (0.000) \\ 0.828\ (0.020) \\ 2.506\ (0.015) \\ 4.184\ (0.020) \\ 5.000\ (0.000) \end{bmatrix}.$$

The estimated intercepts are also plotted in Figure E1, representing the model implied population mean trajectory across measurement occasions. Readers may recall that in Appendix C we have shown that the model implied mean trajectory $\boldsymbol{\tau}$ is equal to y_f when *t* is below μ_1, and is equal to y_c when *t* is above μ_2; it is supported by the estimation results.

The estimated factor loading matrix (with standard error in parenthesis) is

$$\hat{\boldsymbol{\Lambda}} = \left[\begin{array}{c|c} 0.000\ (0.000) & 0.000\ (0.000) \\ -1.400\ (0.017) & -0.278\ (0.005) \\ -0.837\ (0.008) & -0.841\ (0.008) \\ -0.274\ (0.005) & -1.404\ (0.017) \\ 0.000\ (0.000) & 0.000\ (0.000) \end{array}\right].$$

It is usually of less interest to examine and interpret the factor loadings when the focal research question only concerns the estimation of the random knots and their inter-person variability. Nonetheless, here we offer some interpretations of the factor loading matrix for readers who are interested. Each column of the factor loadings matrix is referred to as the *basis curve*, or *basis function*. As demonstrated in Blozis (2004), the basis curves define different aspects of change in the outcome variable across measurement occasions. Individual's growth curve is modeled as a weighted linear combination of the basis curves (Eq. 8), each weighted by the corresponding random coefficient, which may differ from person to person. Therefore, each basis curve can be weighted differently for different individuals, suggesting each aspect of change plays a different role across individuals. Such a weighted linear combination can thus result in the variability of individual growth trajectories. In the first illustrative example, the basis curve with regard to the first random knot is plotted in Figure E2 and the basis curve with regard to the second random knot is plotted in Figure E3. As discussed in the manuscript, the variability of the first random knot and the variability of the second random knot are both significant; therefore, these two basis curves are weighted differently for different individuals when forming a linear combination.

On the other hand, the basis curves are also factor loadings, which can be interpreted as the effect of the latent factors on the measured variables. In the first illustrative example, we have estimated the factor loadings on the first and second random knot, respectively. A close examination reveals that the loadings on the latent factors are zeros before the estimated population mean value of the first random knot ($t < \hat{\mu}_1$) and after the estimated population mean value of the second random knot ($t > \hat{\mu}_2$). Therefore, it implies that at the floor and the ceiling segments, the growth trajectories are modeled as the fixed floor value/ceiling value plus a random measurement error, while the latent factors have no bearing on the measured outcomes. In between the floor and ceiling segments, the factor loadings are all negative, suggesting that at each measurement point, the predicted observed outcome will have a larger value when the individual reaches the transition point earlier than individuals who reach the transition point later. The effect of the first random knot decreases as the measurement time moves further away from the first knot, while the effect of the second random knot increases as the measurement time approaches the second knot.

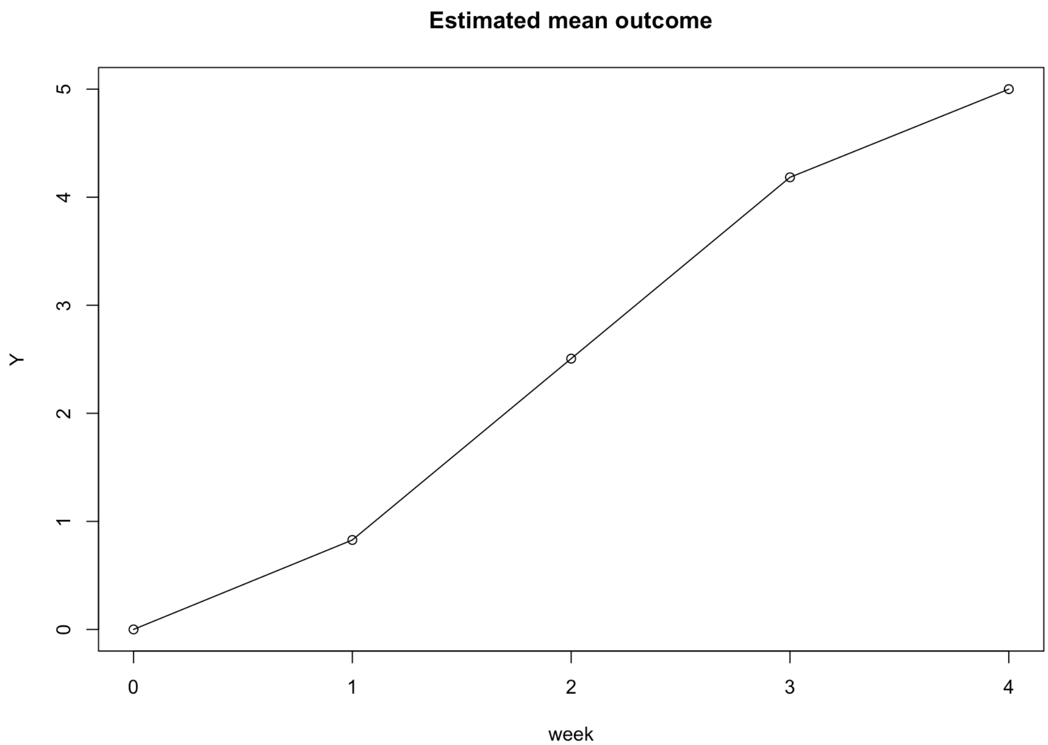

Figure E1. Model implied mean outcome values across measurement occasions.

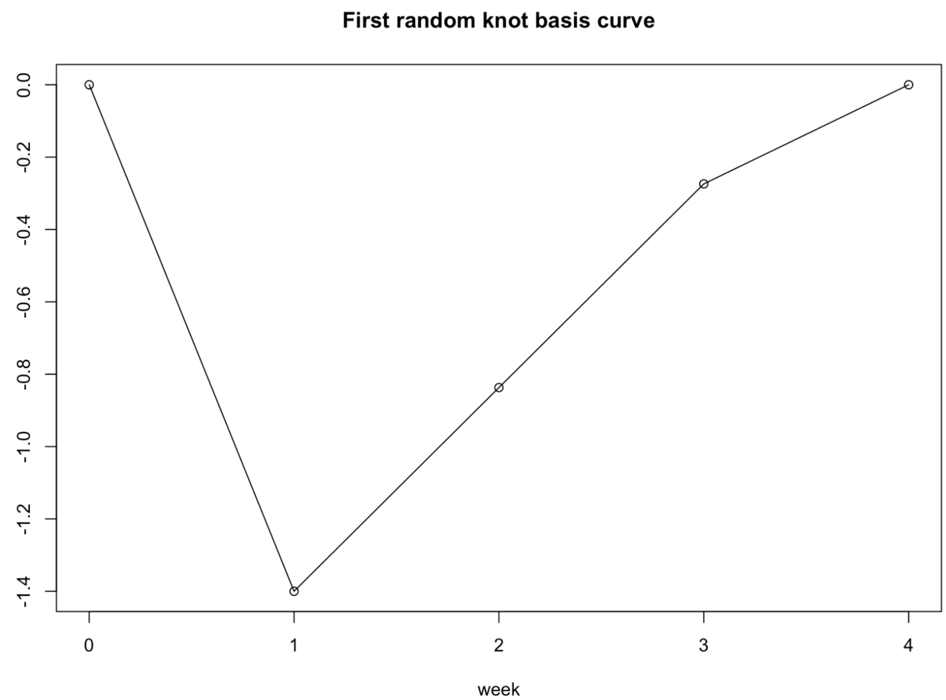

Figure E2. The basis curve with regard to the first random knot (i.e., the factor loadings on the first random knot).

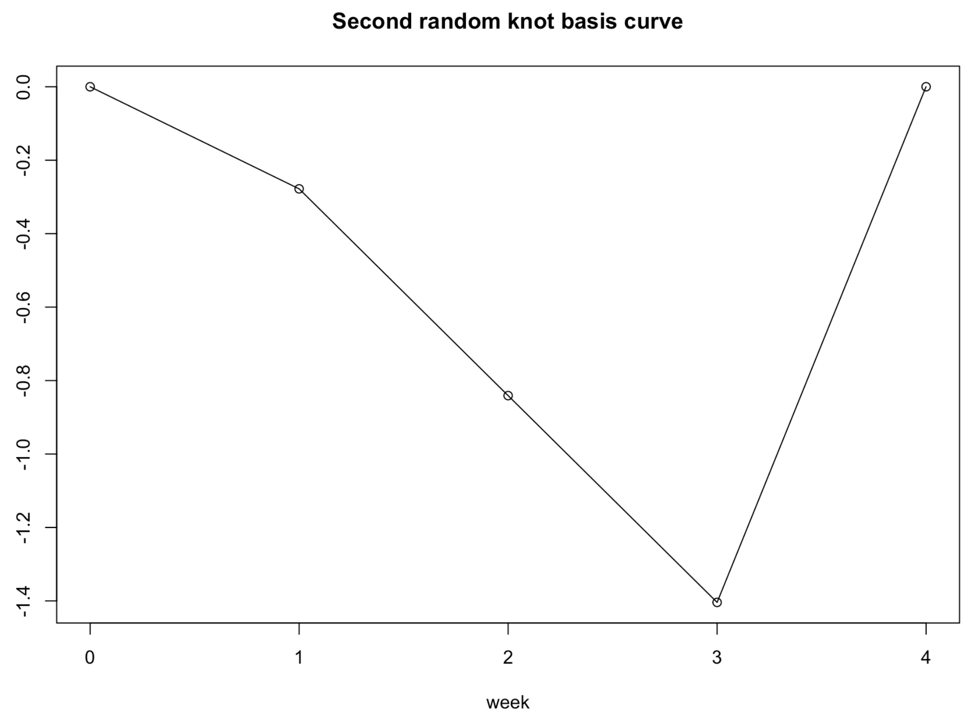

Figure E3. The basis curve with regard to the second random knot (factor loadings on the second random knot factor).

Appendix F

Here we provide the estimated intercept vector, factor loadings matrix, and residual variance-covariance matrix for the second illustrative example. The estimated intercept vector (with standard error in parenthesis) is

$$\hat{\boldsymbol{\tau}} = \begin{bmatrix} 399.323\ (0.736) \\ 399.323\ (0.736) \\ 399.323\ (0.736) \\ 402.513\ (1.126) \\ 418.191\ (0.985) \\ 433.869\ (0.928) \\ 449.548\ (0.972) \\ 465.226\ (1.104) \\ 480.905\ (1.297) \\ 484.964\ (1.103) \end{bmatrix}.$$

The estimated intercepts are plotted in Figure F1, representing the model implied population mean trajectory of PC test scores across school terms. Again, we can find that the model implied mean trajectory $\hat{\boldsymbol{\tau}}$ is equal to $\hat{\mu}_f$ when t is below $\hat{\mu}_1$, and is equal to $\hat{\mu}_c$ when t is above $\hat{\mu}_2$.

The estimated factor loading matrix (with standard error in parenthesis) is

$$\widehat{\boldsymbol{\Lambda}} = \left[\begin{array}{c|c|c|c} 0.000\ (0.000) & 0.000\ (0.000) & 1.000\ (0.000) & 0.000\ (0.000) \\ 0.000\ (0.000) & 0.000\ (0.000) & 1.000\ (0.000) & 0.000\ (0.000) \\ 0.000\ (0.000) & 0.000\ (0.000) & 1.000\ (0.000) & 0.000\ (0.000) \\ -15.094\ (0.443) & -0.584\ (0.202) & 0.963\ (0.013) & 0.037\ (0.013) \\ -12.224\ (0.345) & -3.454\ (0.156) & 0.780\ (0.011) & 0.220\ (0.011) \\ -9.354\ (0.254) & -6.324\ (0.173) & 0.597\ (0.009) & 0.403\ (0.009) \\ -6.484\ (0.179) & -9.915\ (0.240) & 0.414\ (0.009) & 0.586\ (0.009) \\ -3.613\ (0.147) & -12.065\ (0.328) & 0.230\ (0.010) & 0.770\ (0.010) \\ -0.743\ (0.183) & -14.935\ (0.424) & 0.047\ (0.012) & 0.953\ (0.012) \\ 0.000\ (0.000) & 0.000\ (0.000) & 0.000\ (0.000) & 1.000\ (0.000) \end{array}\right].$$

There are four columns of factor loadings, hence four different basis curves, which are plotted in Figures F2-F5, respectively. As discussed in the manuscript, the variability of the first random knot, the random floor, and the random ceiling are all significant; therefore, their corresponding basis curves are weighted differently from child to child when forming a linear combination as the individual growth trajectories.

The basis curves can also be interpreted as factor loadings. At the floor segment ($t < \hat{\mu}_1$), all factor loadings are zero except for the loadings on the random floor latent factor, which are all estimated to be 1. On the other hand, at the ceiling segment ($t > \hat{\mu}_2$), all factor loadings are zero except for the loadings on the random ceiling latent factor, which is estimated to be 1. It thus implies that at the floor and the ceiling segments, the growth trajectories are modeled as the individual floor value/ceiling value plus a random measurement error, while the latent factors of the random knots have no effect on PC test scores. In the middle segment ($\hat{\mu}_1 < t < \hat{\mu}_2$), the factor loadings on the random knots are all estimated to be negative values. It suggests that at each measurement occasion, conditional on other latent factors, individual PC test scores are expected to be higher for children who reach the transition points earlier. Similar with the first illustrative example, the effect of the first random knot decreases as the measurement time moves further away from the first knot, while the effect of the second random knot increases as the measurement time approaches the second knot. The factor loadings on the random floor and random ceiling are estimated to be positive in the middle segment. It thus indicates that at each measurement occasion, conditional on other latent factors, the individual child is expected to have a higher PC test score if he/she has a higher level of floor or ceiling value. The effects of the random floor become smaller as the measurement point is further away from the first knot; in contrast, the random ceiling becomes more important as the measurement point approaches the second knot.

Lastly, as we assumed independent residuals in this illustrative example, the estimated residual variance-covariance matrix is a diagonal matrix

$$\hat{\boldsymbol{\Theta}}_{\boldsymbol{\varepsilon}} = \begin{bmatrix} 306.415\ (45.103) & & & & & & & & & \\ & 199.960\ (26.916) & & & & & & & & \\ & & 162.174\ (20.390) & & & & & & & \\ & & & 148.099\ (24.161) & & & & & & \\ & & & & 257.384\ (27.127) & & & & & \\ & & & & & 303.165\ (30.101) & & & & \\ & & & & & & 213.035\ (22.478) & & & \\ & & & & & & & 80.044\ (13.876) & & \\ & & & & & & & & 84.685\ (19.870) & \\ & & & & & & & & & 51.075\ (27.167) \end{bmatrix}.$$

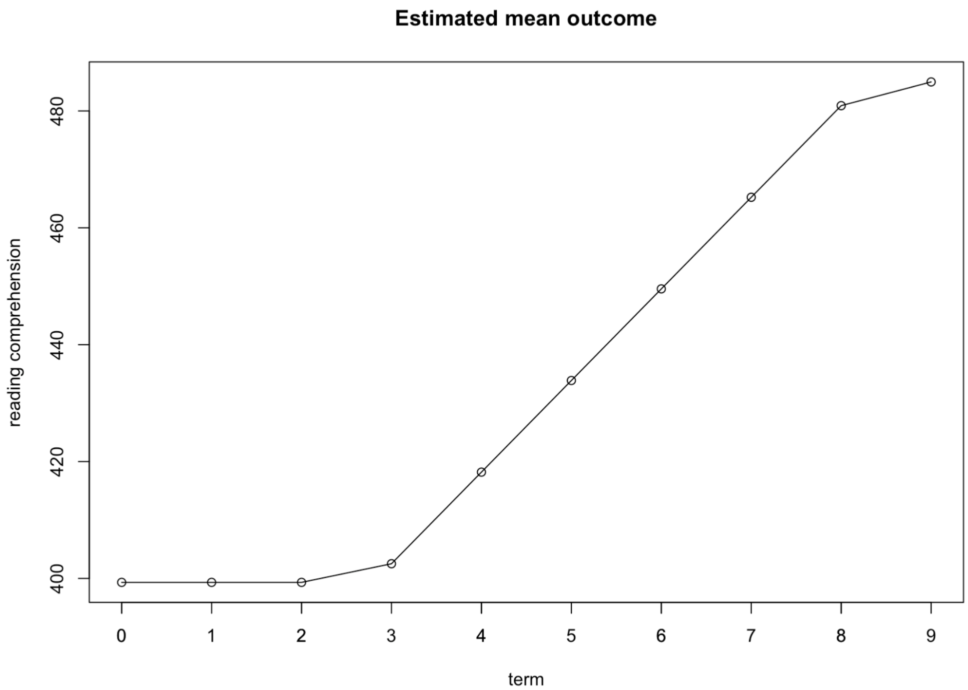

Figure F1. Model implied mean PC test scores across measurement occasions.

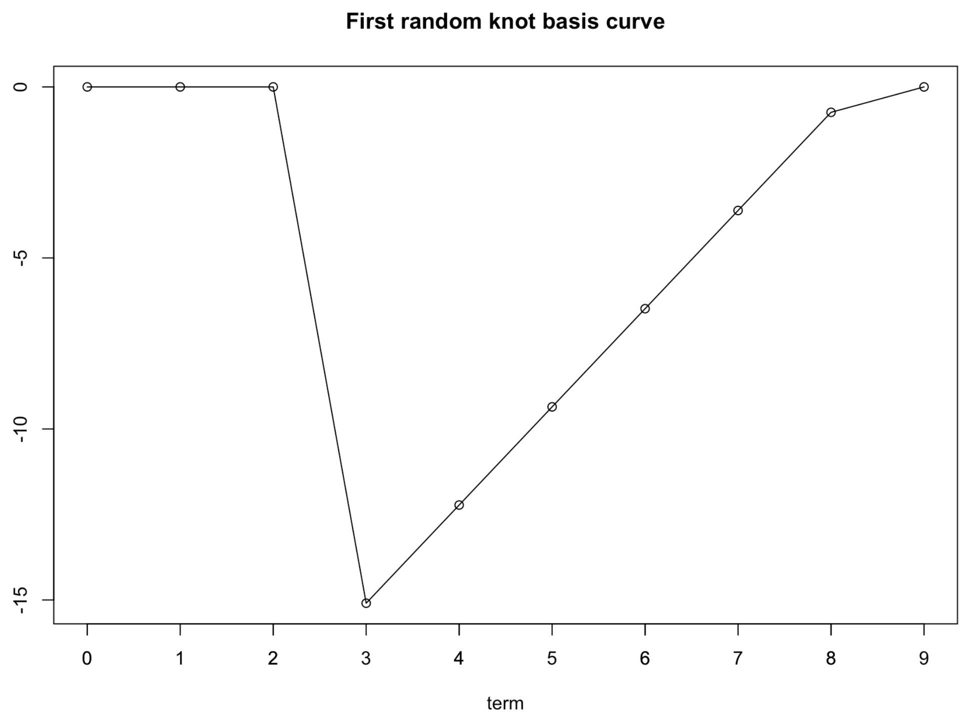

Figure F2. Basis curve with regard to the first random knot (Factor loadings on the first random knot latent factor).

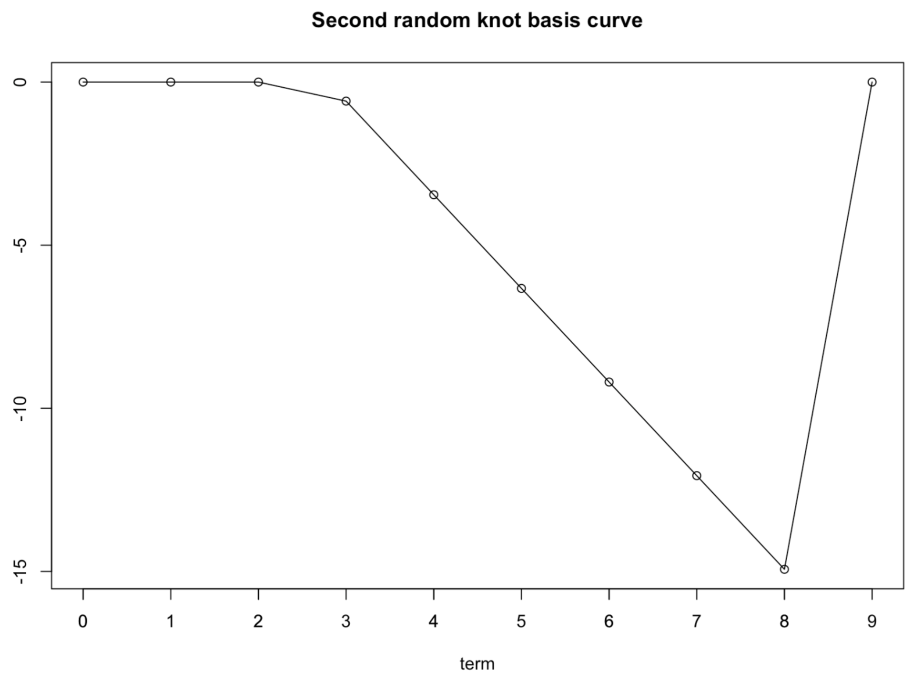

Figure F3. Basis curve with regard to the second random knot (Factor loadings on the second random knot latent factor).

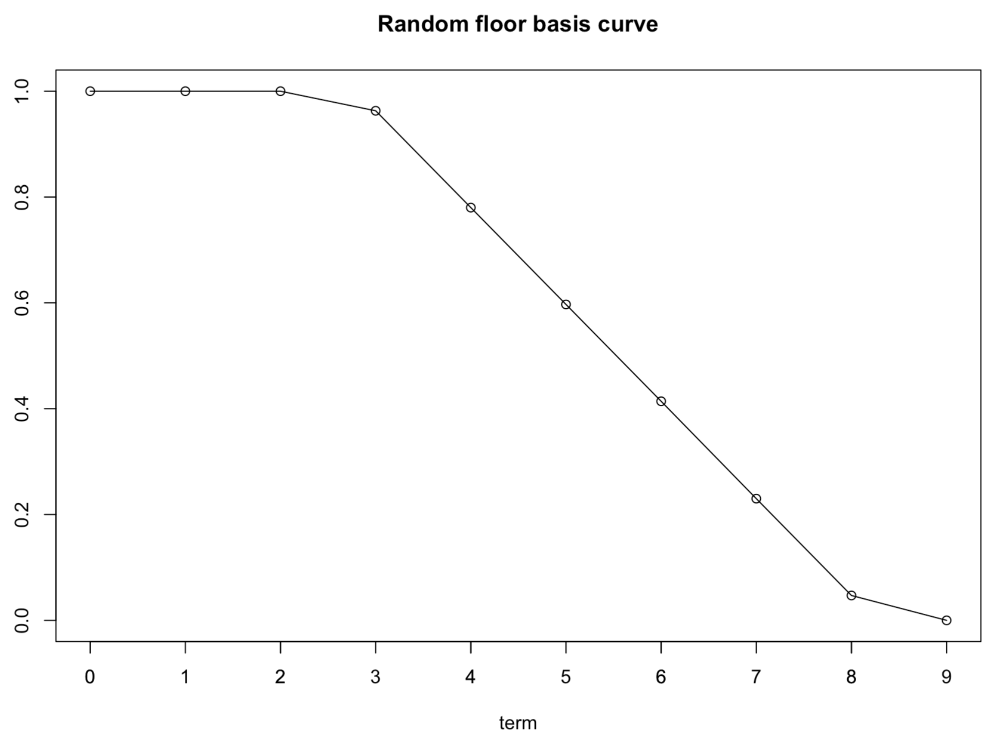

Figure F4. Basis curve with regard to the random floor (factor loadings on the random floor latent factor).

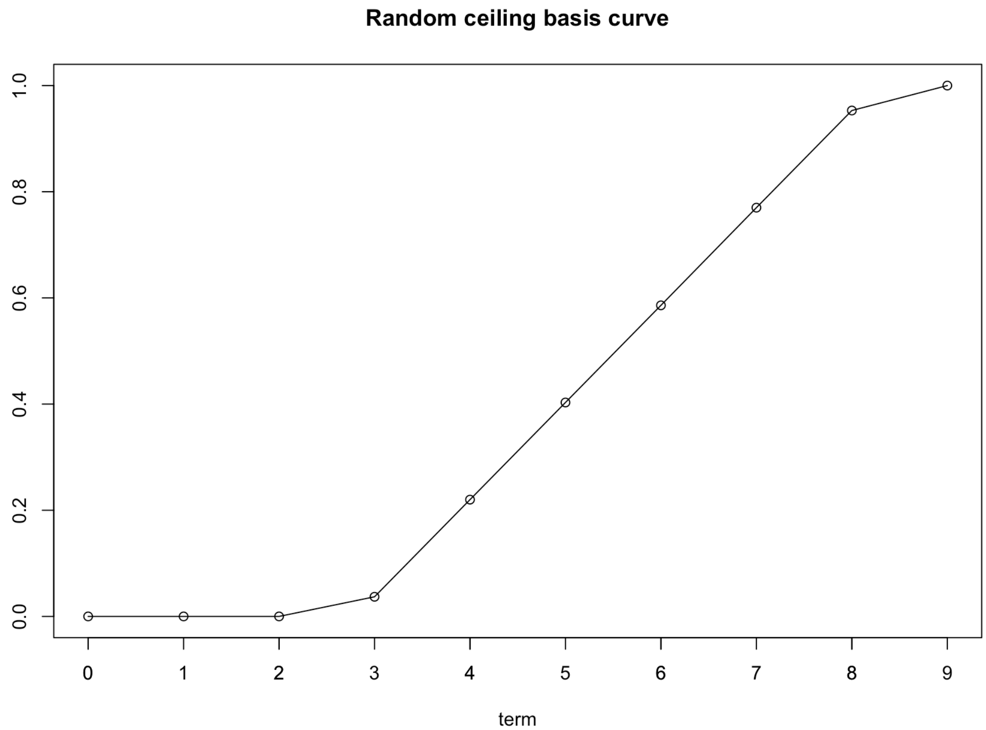

Figure F5. Basis curve with regard to the random ceiling (factor loadings on the random ceiling latent factor).

Appendix G

Here we provide the reparameterization for the piecewise LGM with random knots and fixed floors and ceilings when $\beta < 0$. When the middle component is linearly decreasing, the piecewise model can be written as:

$$g(t,\alpha,\beta) = \begin{cases} g_1(t,\alpha,\beta) = y_c, & t \le \gamma_1 \\ g_2(t,\alpha,\beta) = \alpha + \beta t, & \gamma_1 < t \le \gamma_2 \\ g_3(t,\alpha,\beta) = y_f, & \gamma_2 < t \end{cases}$$

We follow the same procedures to combine the 3 functions into one common target function:

$$g(t,\alpha,\beta) = \text{median}(g_1, g_2, g_3)$$
$$= \text{sum}(g_1, g_2, g_3) - \min(g_1, g_2, g_3) - \max(g_1, g_2, g_3)$$
$$= \text{sum}(g_1, g_2, g_3) - \min(g_2, g_3) - \max(g_1, g_2)$$
$$= (g_1 + g_2 + g_3) - \frac{1}{2}[g_2 + g_3 - \sqrt{(g_2 - g_3)^2}] - \frac{1}{2}[g_1 + g_2 + \sqrt{(g_1 - g_2)^2})]$$
$$= \frac{1}{2}[g_1 + g_3 + \sqrt{(g_2 - g_3)^2} - \sqrt{(g_1 - g_2)^2}]$$
$$= \frac{1}{2}[y_f + y_c + \sqrt{(\alpha + \beta t - y_f)^2} - \sqrt{(y_c - \alpha - \beta t)^2}]$$

Because α, β, γ_1, and γ_2 satisfy the following relations:

$$y_c = \alpha + \beta\gamma_1$$
$$y_f = \alpha + \beta\gamma_2,$$

α and β can thus be expressed as follows:

$$\beta = \left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right]$$
$$\alpha = y_c - \beta\gamma_1$$
$$= y_c - \left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right]\gamma_1$$
$$\alpha = y_f - \beta\gamma_2$$
$$= y_f - \left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right]\gamma_2.$$

Substituting in the target function, we could obtain the reparameterized target function:

$$
\begin{aligned}
y &= f(\gamma_1, \gamma_2, t) \\
&= \frac{1}{2}[y_f + y_c + \sqrt{(\alpha + \beta t - y_f)^2} - \sqrt{(y_c - \alpha - \beta t)^2}] \\
&= \frac{1}{2}\left[y_f + y_c + \sqrt{(y_f - \beta\gamma_2 + \beta t - y_f)^2} - \sqrt{(y_c - y_c + \beta\gamma_1 - \beta t)^2}\right] \\
&= \frac{1}{2}[y_f + y_c + \sqrt{(-\beta\gamma_2 + \beta t)^2} - \sqrt{(\beta\gamma_1 - \beta t)^2}] \\
&= \frac{1}{2}[y_f + y_c + (-\beta)\sqrt{(t - \gamma_2)^2} - (-\beta)\sqrt{(\gamma_1 - t)^2}] \\
&= \frac{1}{2}[y_f + y_c - \beta\sqrt{(t - \gamma_2)^2} + \beta\sqrt{(\gamma_1 - t)^2}] \\
&= \frac{1}{2}\left[y_f + y_c - \left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right]\sqrt{(t - \gamma_2)^2} + \left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right]\sqrt{(\gamma_1 - t)^2}\right].
\end{aligned}
$$

The first-order Taylor series is used to linearize the reparameterized target function:

$$\tilde{y} = f(\gamma_1, \gamma_2, t)|_{\mu_1,\mu_2} + (\gamma_1 - \mu_1)\left.\frac{\partial f}{\partial \gamma_1}\right|_{\mu_1,\mu_2} + (\gamma_2 - \mu_2)\left.\frac{\partial f}{\partial \gamma_2}\right|_{\mu_1,\mu_2}.$$

The following partial first derivatives can be obtained using elementary calculus:

$$
\begin{aligned}
\frac{\partial}{\partial \gamma_1}\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right] &= -\frac{y_c - y_f}{(\gamma_1 - \gamma_2)^2} \\
\frac{\partial}{\partial \gamma_2}\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right] &= \frac{y_c - y_f}{(\gamma_1 - \gamma_2)^2} \\
\frac{\partial}{\partial \gamma_1}\left[\sqrt{(\gamma_1 - t)^2}\right] &= \frac{\gamma_1 - t}{\sqrt{(\gamma_1 - t)^2}} \\
\frac{\partial}{\partial \gamma_2}\left[\sqrt{(t - \gamma_2)^2}\right] &= \frac{\gamma_2 - t}{\sqrt{(\gamma_2 - t)^2}}.
\end{aligned}
$$

The partial first derivatives of the target function with respect to γ_1 and γ_2 can thus be calculated with the basic differentiation as follows:

$$
\begin{aligned}
\frac{\partial f}{\partial \gamma_1} &= \frac{1}{2}\left(0 + 0 - \frac{\partial}{\partial \gamma_1}\left\{\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right]\sqrt{(t - \gamma_2)^2}\right\} + \frac{\partial}{\partial \gamma_1}\left\{\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right]\sqrt{(\gamma_1 - t)^2}\right\}\right) \\
&= \frac{1}{2}\left(\left\{\frac{\partial}{\partial \gamma_1}\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right]\right\}\sqrt{(\gamma_1 - t)^2} + \left\{\frac{\partial}{\partial \gamma_1}\sqrt{(\gamma_1 - t)^2}\right\}\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right] - \left\{\frac{\partial}{\partial \gamma_1}\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right]\right\}\sqrt{(t - \gamma_2)^2}\right) \\
&= \frac{1}{2}\left\{-\frac{y_c - y_f}{(\gamma_1 - \gamma_2)^2}\sqrt{(\gamma_1 - t)^2} + \frac{\gamma_1 - t}{\sqrt{(\gamma_1 - t)^2}}\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right] + \frac{y_c - y_f}{(\gamma_1 - \gamma_2)^2}\sqrt{(t - \gamma_2)^2}\right\}
\end{aligned}
$$

$$= \frac{1}{2}\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right]\left\{\frac{\sqrt{(t-\gamma_2)^2}}{\gamma_1 - \gamma_2} + \frac{\gamma_1 - t}{\sqrt{(\gamma_1 - t)^2}} - \frac{\sqrt{(\gamma_1 - t)^2}}{\gamma_1 - \gamma_2}\right\}$$
$$= \frac{1}{2}\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right]\left\{(\gamma_1 - \gamma_2)^{-1}\left[\sqrt{(t-\gamma_2)^2} - \sqrt{(\gamma_1 - t)^2}\right] + \frac{\gamma_1 - t}{\sqrt{(\gamma_1 - t)^2}}\right\}$$

and

$$\frac{\partial f}{\partial \gamma_2} = \frac{1}{2}\left(0 + 0 - \frac{\partial}{\partial \gamma_2}\left\{\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right]\sqrt{(t-\gamma_2)^2}\right\} + \frac{\partial}{\partial \gamma_2}\left\{\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right]\sqrt{(\gamma_1 - t)^2}\right\}\right)$$
$$= \frac{1}{2}\left(-\left\{\frac{\partial}{\partial \gamma_2}\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right]\right\}\sqrt{(t-\gamma_2)^2} - \left\{\frac{\partial}{\partial \gamma_2}\sqrt{(t-\gamma_2)^2}\right\}\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right] + \left\{\frac{\partial}{\partial \gamma_2}\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right]\right\}\sqrt{(\gamma_1 - t)^2}\right)$$
$$= \frac{1}{2}\left\{-\frac{y_c - y_f}{(\gamma_1 - \gamma_2)^2}\sqrt{(t-\gamma_2)^2} - \frac{\gamma_2 - t}{\sqrt{(\gamma_2 - t)^2}}\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right] + \frac{y_c - y_f}{(\gamma_1 - \gamma_2)^2}\sqrt{(\gamma_1 - t)^2}\right\}$$
$$= \frac{1}{2}\left[\frac{y_c - y_f}{\gamma_1 - \gamma_2}\right]\left\{(\gamma_1 - \gamma_2)^{-1}\left[\sqrt{(\gamma_1 - t)^2} - \sqrt{(t-\gamma_2)^2}\right] + \frac{t - \gamma_2}{\sqrt{(t-\gamma_2)^2}}\right\}.$$

Therefore, the partial first derivatives evaluated at the population means of γ_1 and γ_2 are

$$\left.\frac{\partial f}{\partial \gamma_1}\right|_{\mu_1,\mu_2} = \frac{1}{2}\left[\frac{y_c - y_f}{\mu_1 - \mu_2}\right]\left\{(\mu_1 - \mu_2)^{-1}\left[\sqrt{(t-\mu_2)^2} - \sqrt{(\mu_1 - t)^2}\right] + \frac{\mu_1 - t}{\sqrt{(\mu_1 - t)^2}}\right\}$$
$$\left.\frac{\partial f}{\partial \gamma_2}\right|_{\mu_1,\mu_2} = \frac{1}{2}\left[\frac{y_c - y_f}{\mu_1 - \mu_2}\right]\left\{(\mu_1 - \mu_2)^{-1}\left[\sqrt{(\mu_1 - t)^2} - \sqrt{(t-\mu_2)^2}\right] + \frac{t - \mu_2}{\sqrt{(t-\mu_2)^2}}\right\}.$$

In the final linearized targeted function, the intercept term $\boldsymbol{\tau}$ is:

$$\boldsymbol{\tau} = f(\gamma_1, \gamma_2, \mathbf{t})|_{\mu_1,\mu_2} = \begin{bmatrix} \frac{1}{2}\left[y_f + y_c - \left[\frac{y_c - y_f}{\mu_1 - \mu_2}\right]\sqrt{(0-\mu_2)^2} + \left[\frac{y_c - y_f}{\mu_1 - \mu_2}\right]\sqrt{(\mu_1 - 0)^2}\right] \\ \frac{1}{2}\left[y_f + y_c - \left[\frac{y_c - y_f}{\mu_1 - \mu_2}\right]\sqrt{(1-\mu_2)^2} + \left[\frac{y_c - y_f}{\mu_1 - \mu_2}\right]\sqrt{(\mu_1 - 1)^2}\right] \\ \vdots \\ \frac{1}{2}\left[y_f + y_c - \left[\frac{y_c - y_f}{\mu_1 - \mu_2}\right]\sqrt{(T-\mu_2)^2} + \left[\frac{y_c - y_f}{\mu_1 - \mu_2}\right]\sqrt{(\mu_1 - T)^2}\right] \end{bmatrix}.$$

The factor loading matrix $\boldsymbol{\Lambda}$ are:

$\boldsymbol{\Lambda} =$

$$\begin{bmatrix} \frac{1}{2}\left[\frac{y_c - y_f}{\mu_1 - \mu_2}\right]\left\{(\mu_1-\mu_2)^{-1}\left[\sqrt{(0-\mu_2)^2}-\sqrt{(\mu_1-0)^2}\right]+\frac{\mu_1-0}{\sqrt{(\mu_1-0)^2}}\right\} & \frac{1}{2}\left[\frac{y_c - y_f}{\mu_1 - \mu_2}\right]\left\{(\mu_1-\mu_2)^{-1}\left[\sqrt{(\mu_1-0)^2}-\sqrt{(0-\mu_2)^2}\right]+\frac{0-\mu_2}{\sqrt{(0-\mu_2)^2}}\right\} \\ \frac{1}{2}\left[\frac{y_c - y_f}{\mu_1 - \mu_2}\right]\left\{(\mu_1-\mu_2)^{-1}\left[\sqrt{(1-\mu_2)^2}-\sqrt{(\mu_1-1)^2}\right]+\frac{\mu_1-0}{\sqrt{(\mu_1-0)^2}}\right\} & \frac{1}{2}\left[\frac{y_c - y_f}{\mu_1 - \mu_2}\right]\left\{(\mu_1-\mu_2)^{-1}\left[\sqrt{(\mu_1-1)^2}-\sqrt{(1-\mu_2)^2}\right]+\frac{1-\mu_2}{\sqrt{(1-\mu_2)^2}}\right\} \\ \vdots & \vdots \\ \frac{1}{2}\left[\frac{y_c - y_f}{\mu_1 - \mu_2}\right]\left\{(\mu_1-\mu_2)^{-1}\left[\sqrt{(T-\mu_2)^2}-\sqrt{(\mu_1-T)^2}\right]+\frac{\mu_1-T}{\sqrt{(\mu_1-T)^2}}\right\} & \frac{1}{2}\left[\frac{y_c - y_f}{\mu_1 - \mu_2}\right]\left\{(\mu_1-\mu_2)^{-1}\left[\sqrt{(\mu_1-T)^2}-\sqrt{(T-\mu_2)^2}\right]+\frac{T-\mu_2}{\sqrt{(T-\mu_2)^2}}\right\} \end{bmatrix}.$$

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